The coherency matrix for this case is the same as for the monochromatic wave with components equal to

$$E_x = \sqrt{\langle a_x^2 \rangle}\, e^{j(\omega t + \phi_x)} \tag{7.31}$$

$$E_y = C_1 \sqrt{\langle a_x^2 \rangle}\, e^{j(\omega t + \phi_x + C_2)} \tag{7.32}$$

From (7.21) and (7.30) it is clear that for a completely polarized wave

$$\|J\| = 0$$

Linear Polarization

For linear polarization the wave must, of course, satisfy the requirements for complete polarization, and in addition

$$\phi = 0, \pm\pi, \pm 2\pi, \ldots \tag{7.33}$$

Then the coherency matrices for monochromatic and completely polarized polychromatic waves are, respectively,

$$[J] = \frac{1}{2Z_0}\begin{bmatrix} a_x^2 & (-1)^m a_x a_y \\ (-1)^m a_x a_y & a_y^2 \end{bmatrix} \qquad m = 0, 1, 2, \ldots \tag{7.34}$$

and

$$[J] = \frac{1}{2Z_0}\begin{bmatrix} \langle a_x^2 \rangle & (-1)^m C_1 \langle a_x^2 \rangle \\ (-1)^m C_1 \langle a_x^2 \rangle & C_1^2 \langle a_x^2 \rangle \end{bmatrix} \tag{7.35}$$

More particularly, the matrices

$$[J] = S\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \frac{S}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \frac{S}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7.36}$$

represent linear polarizations that are, respectively, x directed, y directed, 45° from the x axis, and 135° from the x axis.

Circular Polarization

We saw previously that for circular polarization the component amplitudes are equal, and

$$\phi = \pm \tfrac{1}{2}\pi \tag{7.37}$$

for left and right circular polarization, respectively. Then the coherency matrix becomes

$$[J] = \frac{S}{2}\begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}, \frac{S}{2}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix} \tag{7.38}$$

for left and right circular polarization, respectively.

7.4. DEGREE OF POLARIZATION

A plane wave may be considered as the sum of N independent plane waves traveling in the same direction. In particular, we will consider a quasi-monochromatic wave to be the sum of a completely polarized wave and a completely unpolarized wave. We may show that this representation is unique by showing that any coherency matrix can be uniquely expressed in the form

$$[J] = [J^{(1)}] + [J^{(2)}] \tag{7.39}$$

where

$$[J^{(1)}] = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \qquad [J^{(2)}] = \begin{bmatrix} B & D \\ D^* & C \end{bmatrix} \tag{7.40}$$

with

$$A \geqq 0 \qquad B \geqq 0 \qquad C \geqq 0 \qquad BC - DD^* = 0 \tag{7.41}$$

If we compare (7.40) to the special case (7.24), we see that $[J^{(1)}]$ is the coherency matrix for a completely unpolarized wave. If we use $\|J\| = 0$ as the criterion for a completely polarized wave, then $[J^{(2)}]$ is the coherency matrix for a completely polarized wave.

We must next show that the decomposition into completely polarized and completely unpolarized waves is unique. This we will do by obtaining the elements of $[J^{(1)}]$ and $[J^{(2)}]$ from the known elements of $[J]$. From (7.39) and (7.40) we may write

$$A + B = J_{xx} \quad \text{(a)} \qquad D^* = J_{yx} \quad \text{(c)}$$
$$D = J_{xy} \quad \text{(b)} \qquad A + C = J_{yy} \quad \text{(d)} \tag{7.42}$$

Substituting (7.42) into the last equation of (7.41) gives

$$(J_{xx} - A)(J_{yy} - A) - J_{xy}J_{yx} = 0 \tag{7.43}$$

which is a quadratic in A with solution

$$A = \tfrac{1}{2}(J_{xx} + J_{yy}) \pm \tfrac{1}{2}[(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} \tag{7.44}$$

Substituting (7.44) in (7.42a) gives

$$\begin{aligned} B &= \tfrac{1}{2}(J_{xx} - J_{yy}) \mp \tfrac{1}{2}[(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} \\ &= \tfrac{1}{2}(J_{xx} - J_{yy}) \mp \tfrac{1}{2}[(J_{xx} - J_{yy})^2 + 4J_{xy}J^*_{xy}]^{1/2} \end{aligned} \tag{7.45}$$

From the second form of (7.45) we see that the negative sign for the last term is not allowed since it would make B negative, contrary to our hypothesis. Then the A, B, C, D values of (7.42) are found uniquely from

$$\begin{aligned} A &= \tfrac{1}{2}(J_{xx} + J_{yy}) - \tfrac{1}{2}[(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} && \text{(a)} \\ B &= \tfrac{1}{2}(J_{xx} - J_{yy}) + \tfrac{1}{2}[(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} && \text{(b)} \\ C &= \tfrac{1}{2}(J_{yy} - J_{xx}) + \tfrac{1}{2}[(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} && \text{(c)} \\ D &= J_{xy} && \text{(d)} \\ D^* &= J_{yx} && \text{(e)} \end{aligned} \tag{7.46}$$

The Poynting vector magnitude of the total wave is

$$S_t = \mathrm{Tr}\,[J] = J_{xx} + J_{yy} \tag{7.47}$$

and that of the polarized part of the wave is

$$S_p = \mathrm{Tr}\,[J^{(2)}] = B + C = [(J_{xx} + J_{yy})^2 - 4\|J\|]^{1/2} \tag{7.48}$$

Quite reasonably, the ratio of the power densities of the polarized part and the total wave is called the *degree of polarization* of the wave. It is given by

$$R = \frac{S_p}{S_t} = \left[1 - \frac{4\|J\|}{(J_{xx} + J_{yy})^2}\right]^{1/2} \tag{7.49}$$

Now

$$\|J\| \leqq J_{xx}J_{yy} \leqq \tfrac{1}{4}(J_{xx} + J_{yy})^2$$

and therefore

$$0 \leqq R \leqq 1 \tag{7.50}$$

Let us consider the two extreme values of R. For $R = 1$, (7.49) requires that

$$\|J\| = 0$$

which is the condition for complete polarization. Then $|\mu_{xy}| = 1$, and the x and y wave components are mutually coherent. For $R = 0$, (7.49) requires that

$$(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2 = 0$$

which can be satisfied only by

$$J_{xx} = J_{yy} \qquad J_{xy} = J_{yx} = 0$$

It follows that $|\mu_{xy}| = 0$ and E_x and E_y are mutually incoherent.

As we have just seen, $R = 0$ requires that E_x and E_y be mutually incoherent. The converse is not true. For mutual incoherence $J_{xy} = J_{yx} = 0$ and $|\mu_{xy}| = 0$. Then

$$R = \left[1 - \frac{4J_{xx}J_{yy}}{(J_{xx} + J_{yy})^2}\right]^{1/2} = \frac{|J_{xx} - J_{yy}|}{J_{xx} + J_{yy}}$$

We see that $|\mu_{xy}| = 0$ is not sufficient to give an unpolarized wave. To make it completely unpolarized, we must also have

$$J_{xx} = J_{yy}$$

We can separate the matrix $[J]$ of (7.24), representing the unpolarized part of a wave, even further, as

$$[J] = \frac{S}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{S}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.51}$$

which indicates that an unpolarized wave can be regarded as being composed of two independent linearly polarized waves orthogonal to each other, each of equal power density.

Just as readily, we could have written

$$[J] = \frac{S}{4}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix} + \frac{S}{4}\begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix} \tag{7.52}$$

showing that with equal validity we could consider an unpolarized wave to be made up of two independent circular waves of opposite rotation sense.

7.5. STOKES PARAMETERS OF PARTIALLY POLARIZED WAVES

We previously defined the Stokes parameters of a monochromatic wave by the equations†

$$\begin{aligned} S_0 &= |E_x|^2 + |E_y|^2 \quad &(a)\\ S_1 &= |E_x|^2 - |E_y|^2 \quad &(b)\\ S_2 &= 2|E_x|\,|E_y| \cos\phi \quad &(c)\\ S_3 &= 2|E_x|\,|E_y| \sin\phi \quad &(d) \end{aligned} \tag{2.184}$$

For quasi-monochromatic waves a more general definition, which reduces to (2.184) for time-independent amplitude and phase of the wave components, is

$$\begin{aligned} S_0 &= \langle a_x^2\rangle + \langle a_y^2\rangle \quad &(a)\\ S_1 &= \langle a_x^2\rangle - \langle a_y^2\rangle \quad &(b)\\ S_2 &= 2\langle a_x a_y \cos\phi\rangle \quad &(c)\\ S_3 &= 2\langle a_x a_y \sin\phi\rangle \quad &(d) \end{aligned} \tag{7.53}$$

where

$$\phi = \phi_y - \phi_x \tag{7.54}$$

If we compare these parameters to the elements of the coherency matrix (7.14), we see that

$$\begin{aligned} S_0 &= 2Z_0(J_{xx} + J_{yy}) \quad &(a)\\ S_1 &= 2Z_0(J_{xx} - J_{yy}) \quad &(b)\\ S_2 &= 2Z_0(J_{xy} + J_{yx}) \quad &(c)\\ S_3 &= 2Z_0 j(J_{xy} - J_{yx}) \quad &(d) \end{aligned} \tag{7.55}$$

† The author regrets the conflict in notation where S is used for power density and the Stokes parameters. The Stokes parameters will have a numerical subscript and the power density will not.

or, solving for the coherency matrix elements,

$$\begin{aligned} J_{xx} &= \frac{1}{4Z_0}(S_0 + S_1) \quad &(a)\\ J_{yy} &= \frac{1}{4Z_0}(S_0 - S_1) \quad &(b)\\ J_{xy} &= \frac{1}{4Z_0}(S_2 - jS_3) \quad &(c)\\ J_{yx} &= \frac{1}{4Z_0}(S_2 + jS_3) \quad &(d) \end{aligned} \tag{7.56}$$

In terms of the Stokes parameters, the statement

$$\|J\| \geqq 0 \tag{7.22}$$

becomes

$$S_0^2 \geqq S_1^2 + S_2^2 + S_3^2 \tag{7.57}$$

For a completely polarized wave the requirement

$$\|J\| = 0$$

gives immediately

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

in accordance with (2.185).

Just as we separated the coherency matrix of a quasi-monochromatic wave into the sum of coherency matrices for a completely polarized wave and a completely unpolarized wave, we can decompose a wave in the same manner in terms of its Stokes parameters. We write for the general wave

$$\begin{aligned} S_0 &= S_0^{(1)} + S_0^{(2)} \quad (a) \qquad & S_2 &= S_2^{(1)} + S_2^{(2)} \quad (c)\\ S_1 &= S_1^{(1)} + S_1^{(2)} \quad (b) \qquad & S_3 &= S_3^{(1)} + S_3^{(2)} \quad (d) \end{aligned} \tag{7.58}$$

where superscript (1) refers to a completely unpolarized wave and (2) to the polarized wave.

Unpolarized Waves

For a completely unpolarized wave we found earlier that

$$J_{xx} = J_{yy} \quad (a) \qquad J_{xy} = J_{yx} = 0 \quad (b) \tag{7.23}$$

Then from (7.55) we have

$$S_1^{(1)} = S_2^{(1)} = S_3^{(1)} = 0 \qquad (7.59)$$

Complete Polarization

For this case we have, rewriting (2.185),

$$(S_0^{(2)})^2 = (S_1^{(2)})^2 + (S_2^{(2)})^2 + (S_3^{(2)})^2 \qquad (7.60)$$

Degree of Polarization

In light of (7.59) the general Stokes parameters of (7.58) simplify to

$$\begin{aligned} S_0 &= S_0^{(1)} + S_0^{(2)} \quad (a) \qquad S_2 = S_2^{(2)} \quad (c) \\ S_1 &= S_1^{(2)} \qquad\qquad (b) \qquad S_3 = S_3^{(2)} \quad (d) \end{aligned} \qquad (7.61)$$

Equations (7.60) and (7.61) can be combined to give

$$S_0^{(1)} = S_0 - \sqrt{S_1^2 + S_2^2 + S_3^2} \qquad (7.62)$$

and

$$S_0^{(2)} = \sqrt{S_1^2 + S_2^2 + S_3^2} \qquad (7.63)$$

The degree of polarization was defined earlier as the ratio of power densities of the polarized part and the total wave. But $S_0^{(2)}$ measures the density of the polarized part and S_0 the density of the total wave. Then the degree of polarization of the wave in terms of its Stokes parameters is

$$R = \frac{S_0^{(2)}}{S_0} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \qquad (7.64)$$

7.6. POLARIZATION RATIO OF PARTIALLY POLARIZED WAVES

We can obtain the polarization ratio and the polarization ellipse characteristics of the polarized part of a wave just as we did for the completely polarized wave. From (2.192a) and the relation between p and P, we can write the polarization ratio in terms of the Stokes parameters for the polarized part of the wave as

$$P = \frac{S_2^{(2)} + jS_3^{(2)}}{S_0^{(2)} + S_1^{(2)}} \qquad (7.65)$$

and if we use (7.63) and (7.61) to find P in terms of the parameters of the total wave, this becomes

$$P = \frac{S_2 + jS_3}{S_1 + \sqrt{S_1^2 + S_2^2 + S_3^2}} \qquad (7.66)$$

In the same way the circular polarization ratio q, from (2.192b), is

$$q = \frac{S_1^{(2)} - jS_2^{(2)}}{S_0^{(2)} - S_3^{(2)}} = \frac{S_1 - jS_2}{\sqrt{S_1^2 + S_2^2 + S_3^2} - S_3} \qquad (7.67)$$

In terms of the coherency matrix elements for the partially polarized wave, the polarization ratio becomes, substituting in (7.66) from (7.55) and making use of (7.64),

$$P = \frac{2J_{yx}}{(R+1)J_{xx} + (R-1)J_{yy}} \qquad (7.68)$$

where R is the degree of polarization of the wave.

For complete polarization we have

$$R = 1 \qquad J_{xx} = \frac{1}{2Z_0} E_x E_x^* \qquad J_{yx} = \frac{1}{2Z_0} E_x^* E_y$$

and (7.68) reduces to

$$P = \frac{E_y}{E_x}$$

7.7. RECEPTION OF PARTIALLY POLARIZED WAVES

A wave with field intensity **E** falling on a receiving antenna with effective length **h** produces an open-circuit voltage at the antenna terminals,

$$V = \mathbf{E} \cdot \mathbf{h} \qquad (3.15)$$

This holds whether **E** is coherent or not, but we are concerned here with partially polarized waves and will accordingly consider the power supplied to a matched load on the antenna to be [5]

$$W = \frac{\langle VV^* \rangle}{8R_a} \qquad (7.69)$$

where R_a is the antenna resistance (radiation resistance plus loss resistance).

Using (A.5) and (A.8), R_a may be put into the form

$$R_a = \frac{Z_0 \mathbf{h}\cdot\mathbf{h}^*}{4A_e} \tag{7.70}$$

and if we use this and (3.15) in (7.69), we get

$$W = \frac{A_e}{2Z_0 \mathbf{h}\cdot\mathbf{h}^*} \langle (\mathbf{E}\cdot\mathbf{h})(\mathbf{E}\cdot\mathbf{h})^* \rangle \tag{7.71}$$

If we note that time averaging is unnecessary for the receiving antenna, the received power becomes

$$W = \frac{A_e}{2Z_0 \mathbf{h}\cdot\mathbf{h}^*} (|h_x|^2 \langle E_x E_x^* \rangle + h_x h_y^* \langle E_x E_y^* \rangle + h_x^* h_y \langle E_x^* E_y \rangle + |h_y|^2 \langle E_y E_y^* \rangle) \tag{7.72}$$

which becomes, using the elements of the coherency matrix of the incident wave,

$$W = \frac{A_e}{\mathbf{h}\cdot\mathbf{h}^*} (|h_x|^2 J_{xx} + h_x h_y^* J_{xy} + h_x^* h_y J_{yx} + |h_y|^2 J_{yy}) \tag{7.73}$$

We saw earlier that a partially polarized plane wave may be considered the sum of a completely polarized wave and a completely unpolarized wave. The coherency matrix elements of the component waves are given by (7.40). Substituting into the equation for received power then gives

$$W = \frac{A_e}{\mathbf{h}\cdot\mathbf{h}^*} [|h_x|^2 (A+B) + h_x h_y^*(D) + h_x^* h_y (D^*) + |h_y|^2 (A+C)] \tag{7.74}$$

This form may be separated to give

$$W = W' + W'' = A_e A + \frac{A_e}{\mathbf{h}\cdot\mathbf{h}^*} (|h_x|^2 B + h_x h_y^* D + h_x^* h_y D^* + |h_y|^2 C) \tag{7.75}$$

where the first term,

$$W' = A_e A \tag{7.76}$$

which represents the power received from the unpolarized portion of the wave, is independent of the polarization characteristics of the receiving antenna. It is informative to express this power in terms of the degree of polarization of the wave. From (7.46a) and (7.49) we get

$$A = \tfrac{1}{2}(J_{xx} + J_{yy})(1-R) \tag{7.77}$$

or, in terms of the power density of the wave,

$$W' = A_e \tfrac{1}{2} S_t (1-R) \tag{7.78}$$

Note that if the wave is unpolarized ($R = 0$), the maximum power that can be extracted from the wave is one-half the power that could be utilized from a completely polarized wave polarization matched to the receiving antenna.

We need not be concerned further with W' since nothing we can do with the polarization of the receiving antenna will either increase or decrease it. We therefore turn our attention to the power received from the completely polarized part of the wave and attempt to maximize it. From (7.41) and (7.46) we note that B and C in

$$W'' = \frac{A_e}{\mathbf{h}\cdot\mathbf{h}^*} (|h_x|^2 B + h_x h_y^* D + h_x^* h_y D^* + |h_y|^2 C) \tag{7.79}$$

are positive real. We therefore first maximize the sum of the two middle terms of (7.79) by setting

$$h_x = |h_x| e^{j\beta_x} \quad \text{(a)} \qquad h_y = |h_y| e^{j\beta_y} \quad \text{(b)} \qquad D = |D| e^{j\delta} \quad \text{(c)} \tag{7.80}$$

It is at once obvious that the sum

$$h_x h_y^* D + h_x^* h_y D^*$$

is maximum if we choose

$$\beta_y - \beta_x = \delta \tag{7.81}$$

Then W'' becomes

$$W''_m = \frac{A_e}{\mathbf{h}\cdot\mathbf{h}^*} (|h_x|^2 B + 2|h_x|\,|h_y|\,|D| + |h_y|^2 C) \tag{7.82}$$

We can maximize W''_m by varying $|h_x|$ and $|h_y|$ while holding $\mathbf{h}\cdot\mathbf{h}^*$ constant. This is an appropriate constraint and was discussed in Section 3.4. Differentiating W''_m with respect to $|h_x|$ given by

$$|h_x| = (\mathbf{h}\cdot\mathbf{h}^* - |h_y|^2)^{1/2} \tag{7.83}$$

and setting the derivative to zero gives

$$\frac{|h_y|^2 - |h_x|^2}{|h_x|\,|h_y|} = \frac{C-B}{|D|} \tag{7.84}$$

with solution

$$\frac{|h_y|}{|h_x|} = \frac{C}{|D|} \quad \text{(a)} \qquad \frac{|h_x|}{|h_y|} = \frac{B}{|D|} \quad \text{(b)} \tag{7.85}$$

Note that $|D| \neq 0$ unless the wave is completely unpolarized. Obviously one of these forms is the inverse of the other, and this leads to the requirement

$$BC - |D|^2 = 0$$

which agrees with (7.41). Since $|D| \neq 0$, then $B \neq 0$ and $C \neq 0$.

Combining (7.85) and (7.81) leads to the relations

$$\frac{h_y}{h_x} = \frac{C}{D^*} \quad \text{(a)} \qquad \frac{h_x}{h_y} = \frac{B}{D} \quad \text{(b)} \tag{7.86}$$

If these values are substituted into the equation for W''_m, the power received from the polarized part of the wave becomes

$$W''_{mm} = A_e(B + C) \tag{7.87}$$

which is obviously maximum power rather than minimum.

From the equations for B and C, (7.46); the power densities (7.47) and (7.48); and the definition for degree of polarization, R; the maximum power that may be received from the polarized part of the wave is

$$W''_{mm} = A_e S_p = A_e S_t R \tag{7.88}$$

where S_p is the power density of the polarized part of the wave and S_t is that of the total wave.

It may be shown that if the wave is completely polarized, the choices made for the receiving antenna effective lengths in (7.86) are the same as those made in (3.36). This is left as an exercise.

It was noted earlier that for the unpolarized part of the wave the maximum power that can be received is one-half the power that could be received from a polarized wave of the same power density using a matched polarization receiver. The received power is independent of the receiver polarization. Then in order to maximize total received power, we need only to match our receiver to the completely polarized portion of the wave using (7.86). The total received power is then the sum

$$W_m = W' + W''_{mm} = \tfrac{1}{2} A_e S_t (1 + R) \tag{7.89}$$

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PROBLEMS

7.1. Consider a monochromatic wave so that, for example, in (7:79), $B = J_{xx}$, and the coherency matrix elements can be simplified. Show that the choices made for the receiving antenna in (7.86) reduce to the choices made for the monochromatic wave in (3.36).

7.2. Derive Eq. (7.87).

7.3. Obtain the effective length components analogous to those in (7.86) if it is desired to minimize the power received from the polarized part of the wave.